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Comparison of the Trapezoidal Rule and Simpson's Rule in the Riemann-Liouville Fractional Integral Approach

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Abstract

Research on calculus has developed a lot, including fractional calculus. Fractional calculus is a branch of mathematics that transforms the orders of derivatives and integrals into rational or even real values. In finding the value of the derivative and the fractional integral, a numerical method is needed to find it, because of the difficulty if it is done using an analytical method. In this paper, we will describe the Riemann-Liouville fractional integral approach using the trapezoidal rule and Simpson's rule. We also provide an overview of the comparisons and errors that result from the two methods.

Keywords: Fractional calculus, Riemann-Liouville fractional integral, trapezoidal rule, Simpson’s rule

1. Introduction

The growth of orders on integrals and derivatives of integers into real and complex numbers is studied in fractional calculus, a field of mathematical analysis. Fractional derivatives are employed in a wide range of applications. Among them are engineering (Loverro, 2004), biology (Ghanbari et al., 2020), biotechnology (Magin, 2004), health (Iyiola & Zaman, 2014), dan other specialties. Because fractional derivatives have such a huge impact, they have gotten a lot of attention in recent years, which could be because they have a broader reach than traditional integer derivatives (Robinson, 1981).

When analytical integration is hard to evaluate, numerical integration can be a useful tool for generating precise integral estimates. The trapezoidal rule and Simpson's rule, as shown in (Yeh, 2002; Ling et al., 2017), are two ways for estimating the integral value. Numerical integration for fractional integrals can be utilized to solve problems with fractional derivatives, according to existing research. At (Diethelm et al., 2002; Diethelm et al., 2004; Baskonus & Bulut, 2015), the Adams-Bashforth-Moulton method for solving the fractional differential equation has been examined. According to Odibat (2006) discusses the modified trapezoidal rule technique for the fractional integral technique, as well as Caputo's fractional derivative with error formula. In addition, Pandiangan et al. According to Pandiangan et al. (2021) employ a modified trapezoidal technique to approximate the Caputo fractional derivative by replacing α with $-\alpha$. Kumar et al. (2019) use quadratic and cubic approaches to solve the integral approach of the Rieman Liouville fraction and the derivative of the Caputo fraction.

In this paper, we present the Riemann-Liouville fractional integral approximation method with a modified trapezoidal rule and Simpson's rule based on the previously mentioned research. We also provide examples for several integral forms and their error ratio analysis.

2. Materials and Methods

2.1. Materials

Using the trapezoidal and Simpson's rules, we shall explain the Riemann-Liouville fractional integral technique in this paper. We also give an outline of the contrasts and mistakes caused by the two approaches.

2.2. Methods

2.2.1. Riemann-Liouville Fractional Integral

Combining the derivative and integral of the order of integers yields the Riemann-Liouville integral. First, to obtain the Cauchy function, we must generalize the definition of the integral. If $f(\tau)$ can be integrated on any interval (a, t), then integral

$$f^{-1}(t) = \int_a^t f(\tau)\mathrm{d}\tau$$

exist. Then for two integrals:

$$f^{-2}(t) = \int_a^t \mathrm{d}\tau_1 \int_a^{\tau_1} f(\tau)\mathrm{d}\tau = \int_a^t f(\tau)\mathrm{d}\tau \int_a^t \mathrm{d}\tau_1$$
$$= \int_a^t (t-\tau) f(\tau)\mathrm{d}\tau.$$

If the last equation is integrated, then we get three integrals of $f(\mathrm{t})$

$$f^{-3}(t) = \int_a^t \mathrm{d}\tau_1 \int_a^{\tau_1} \mathrm{d}\tau_2 \int_a^{\tau_2} f(\tau)\mathrm{d}\tau = \int_a^t \mathrm{d}\tau_1 \int_a^t (t-\tau) f(\tau)\mathrm{d}\tau.$$
$$= \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)\mathrm{d}\tau.$$

Then, we can get the Cauchy formulation

$$f^{-n}(t) = \frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau. \tag{1}$$

If n in Cauchy's formula in equation (1) is replaced by a real number p, we will get an integral for any order.

Definition 2.1 [14] Riemann-Liouville fractional integral with order $\alpha > 0$ and $t > 0$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

Then the Riemann-Liouville fractional derivative with order $\alpha > 0$ defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n t}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau\,,\ n-1 < \alpha \le n$$

where n is an integer.

The following are some instances of Riemann-Liouville fractional integral results for $\alpha, \beta > 0$, $t > 0$, and $\gamma > -1$:

$$J^\alpha J^\beta = J^{\alpha+\beta},$$
$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} x^{\gamma+\alpha},$$
$$J^\alpha e^{at} = t^\alpha \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\alpha+k+1)},$$
$$J^\alpha \cos(at) = t^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k (at)^k}{\Gamma(\alpha+2k+1)},$$
$$J^\alpha \sin(at) = t^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k (at)^k}{\Gamma(\alpha+2k+2)}.$$

2.2 Trapezoidal Rule in Numerical Integration

Based on [6], the trapezoidal rule is one of the numerical methods used in estimating the integral or area under the curve. For example, given the definite integral

$$I(x) = \int_a^b f(x)\,dx. \tag{2}$$

The trapezoidal rule replaces $f(x)$ with a linear polynomial

$$P_1(x) = \frac{(b-x)f(a) + (x-a)f(b)}{b-a}$$

Which interpolates $f(x)$ in a and b so that integral (2) can be approximated by the integral of from $P_1(x)$ in $[a,b]$ which is given by

$$T_1(f) = (b-a)\left[\frac{f(a)+f(b)}{2}\right]. \tag{3}$$

To improve $T_1(f)$ in (3), $[a,b]$ can be divided tasks into specific subintervals, with each subinterval using approach (3) to derive the trapezoid rule formula for n subinterval

$$T_n(f) = h\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right], \tag{4}$$

where $h = (b-a)/n$ is the length of each subinterval and the vertex of integration $x_j = a + jh$ for $j = 0,1,\dots,n$. Form (4) can be generally be written as

$$T_n(f) = \frac{h}{2}\big(f(a) + f(b)\big) + h\sum_{k=1}^{n-1} f(x_k).$$

2.3 Simpson's Rule in Numerical Integration

Another method for estimating integral values is to use Simpson's Rule. In Simpson's Rule, the $T_1(f)$ approximation in (3) is improved by using quadratic interpolation to approximate $f(x)$ in $[a,b]$. Assume $P_2(x)$ is a quadratic polynomial that interpolates $f(x)$ on the variables a, $c = (a+b)/2$, and b. We can approximate $I(f)$, with this

$$I(f) \approx \int_a^b \left[\frac{(x-c)(x-b)}{(a-c)(a-b)}f(a) + \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c) + \frac{(x-a)(x-c)}{(b-a)(b-c)}f(b)\right]dx. \tag{5}$$

This integral can be computed directly, but it is more convenient to first introduce $h = (b-a)/2$ and then substitute the integral's variables to obtain the evaluation result of (5)

$$S_2(f) = \frac{h}{3}\left[\frac{1}{2}f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{6}$$

To get a more accurate result for the interval $[a,b]$, assume n to an even integer, $h = (b-a)/n$, and define the evaluation point for $f(x)$ as

$$x_j = a + jh, \qquad j = 0,1,\dots,n.$$

The interval $[a,b] = [x_0, x_n]$ is divided into three subintervals, each with three assessment points, resulting in the following the formula for Simpson's rule

$$S_n(f) = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]. \quad (7)$$

3. Results and Discussion

3.1. Modified Trapezoidal Rule

In this section, we will use a generalization of the trapezoidal rule to approximate the fractional integral $J^\alpha f(x)$ in order α.

Theorem 3.1 [12] Assume the interval $[0, a]$ is divided into k subintervals $[x_j, x_{j+1}]$ with the same width $h = a/k$ with point $x_j = jh$, for every $0,1, \dots, k$. Modified trapezoidal rule:

$$T(f,h,\alpha) = ((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha)\frac{h^\alpha f(0)}{\Gamma(\alpha+2)} + \frac{h^\alpha f(a)}{\Gamma(\alpha+2)} + \sum_{j=1}^{k-1}((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1})\frac{h^\alpha f(x_j)}{\Gamma(\alpha+2)}. \quad (8)$$

is an approximation to the fractional integral

$$(J^\alpha f(x))(a) = T(f,h,\alpha) - E_T(f,h,\alpha), \;\; a > 0, \alpha > 0.$$

The modified trapezoidal rule is used to approximate various forms of fractional integrals in the following examples.

Table 1: The result of the fractional integral $\left(J^{0.5} f(x)\right)(1)$ with $f(x) = \sin x$ using the modified trapezoid rule

k	h	$T(f,h,\alpha)$	$Error$	Ratio
10	0.1	0.6691782501	5.06×10^{-4}	
20	0.05	0.6695538553	1.30×10^{-4}	3.89
40	0.025	0.6696509910	3.32×10^{-5}	3.91
80	0.0125	0.6696758212	8.44×10^{-6}	3.93
160	0.00625	0.6696820649	2.19×10^{-6}	3.85

Table 2: The result of the fractional integral $\left(J^{0.5} f(x)\right)(1)$ with $f(x) = e^x$ using the modified trapezoid rule

k	h	$T(f,h,\alpha)$	$Error$	Ratio
10	0.1	2.292437779	1.74×10^{-3}	
20	0.05	2.291145923	4.48×10^{-4}	3.88
40	0.025	2.290812410	1.14×10^{-4}	3.92
80	0.0125	2.290727055	2.88×10^{-5}	3.96
160	0.00625	2.290705459	7.21×10^{-6}	3.99

3.2 Simpson's Rule for Fractional Integral

Theorem 3.2 Blaszczyk & Siedlecki, (2014) Assume the interval $[0, b]$ divided into k subintervals $[t_i, t_{i+1}]$ with a constant distance $h = b/k$ and nodes $t_i = ih$ for any $i = 0,1, \dots, k$. Simpson's rule for the fractional integral $J^\alpha f(t)$ is as follows:

$$S(t_i,h,\alpha)=\frac{h^\alpha}{2\Gamma(\alpha)}\sum_{j=0}^{\frac{i-2}{2}}\Big\{\left[f_{2j}-2f_{2j+1}+f_{2j+2}\right]\left[\frac{i^2}{\alpha}c_{i,j}^{\alpha}-\frac{2i}{\alpha+1}c_{i,j}^{\alpha+1}+\frac{1}{\alpha+2}c_{i,j}^{\alpha+2}\right]$$
$$-\left[(4j+3)f_{2j}-(8j+4)f_{2j+1}+(4j+1)f_{2j+2}\right]\left[\frac{i}{\alpha}c_{i,j}^{\alpha}-\frac{1}{\alpha+1}c_{i,j}^{\alpha+1}\right]$$
$$-\left[(2j+1)(2j+2)f_{2j}-4j(2j+2)f_{2j+1}+2j(2j+1)f_{2j+2}\right]\frac{i}{\alpha}c_{i,j}^{\alpha}\Big\} \qquad (9)$$

when $f_n=f(t_n)=f(nh)$ and $c_{i,j}^{\beta}=(i-2j)^\beta-(i-2j-2)^\beta$ .For $\alpha=1$ formula (9) has a simpler form

$$\int_{t_0}^{t_i} f(\tau)d\tau \approx S(t_i,h,1)=\frac{h}{3}\sum_{j=0}^{\frac{i-2}{2}}\left(f_{2j}+4f_{2j+1}+f_{2j+2}\right).$$

In the following, we give some uses of Simpson's rule to approximate the fractional integral for several different integrals.

Table 3: The result of the fractional integral $\left(J^{0.5}f(x)\right)(1)$ with $f(x)=\sin x$ using Simpson's rule

k	H	$S(t_i,h,\alpha)$	$Error$	Ratio
10	0.1	0.6696793673	4.89×10^{-6}	
20	0.05	0.6696838268	4.33×10^{-7}	11.29
40	0.025	0.6696842213	3.83×10^{-8}	11.30
80	0.0125	0.6696842562	3.39×10^{-9}	11.29
160	0.00625	0.6696842593	3.00×10^{-10}	11.30

Table 4: The result of the fractional integral $\left(J^{0.5}f(x)\right)(1)$ with $f(x)=e^x$ using Simpson's rule

k	h	$S(t_i,h,\alpha)$	$Error$	Ratio
10	0.1	2.290717870	1.96×10^{-5}	
20	0.05	2.290700127	1.87×10^{-6}	10.48
40	0.025	2.290698427	1.74×10^{-7}	10.74
80	0.0125	2.290698268	1.59×10^{-8}	10.94
160	0.00625	2.290698254	1.44×10^{-9}	11.04

3.3 Result Analysis

The multiplication of k is used in the calculation results for Table 1 to Table 4. Because the value of the function used in the following calculation includes all the values of the function used in the previous calculation, this multiplication is done to simplify the calculation. The ratio column in Table 1 and Table 2 has a value of roughly 4 for each k value in the table, indicating that the error will drop by 4 times from the original mistake for every doubling of k. Meanwhile, Table 3 and Table 4 show that the ratio column is worth roughly 11 for each value of k in the table, implying that doubling k reduces the error by 11 times. This value is higher than the trapezoid rule's successive error ratio. Furthermore, the error value of the result with Simpson's rule for each integral value is smaller than the trapezoidal rule, as shown in Table 1 to Table 4. For each integral form, Simpson's rule is a better approximation than the trapezoidal method. Furthermore, consider Table 5 and Table 6 below:

Table 5: The result of the fractional integral $\left(J^{1}f(x)\right)(2\pi)$ with $f(x)=1/(2+\cos x)$ using the modified trapezoidal rule

k	T	$Error$	Rasio
2	4.188790	5.61×10^{-1}	
4	3.665191	3.76×10^{-2}	14.928
8	3.627792	1.93×10^{-4}	194.7772
16	3.627599	*	*
32	3.627599	*	*

Table 6: The result of the fractional integral $\left(J^{0.5}f(x)\right)(2\pi)$ with $f(x) = 1/(2+\cos x)$ using the modified trapezoidal rule

k	$T(f,h,\alpha)$	$Error$	Rasio
2	1.679189	2.26×10^{-1}	
4	1.480484	2.74×10^{-2}	8.2589
8	1.457513	4.40×10^{-3}	6.2171
16	1.454273	1.16×10^{-3}	3.7859
32	1.453415	3.05×10^{-4}	3.8131

The function $f(x) = 1/(2+\cos x)$ in Table 3 and Table 4 is periodic with a period of 2π. Based on Weideman, (2002) this form of function is an example of geometric convergence. The integral form (2) returns the precise number of significant digits approximately doubled by doubling the value of k, therefore convergence is fast. For fractional integrals, however, this is not always the case.

For example, Table 5 explains that the error ratio of the integral $\left(J^{\alpha}\left(1/(2+\cos x)\right)\right)(2\pi)$ with $\alpha = 1$ change happens more quickly. The result of integral $k = 16$ and $k = 32$ was correct up to the limits due to computational errors. Table 6 indicates that for $\alpha = 0.5$, the error ratio of the same integral does not vary as quickly as the ratio in Table 5. This indicates that for $\alpha = 1$, the modified trapezoidal rule for the fractional integral of the periodic function converges quickly, but this is not always the case for other alpha values.

4. Conclussion

In this paper, we explain how to approach the Riemann-Liouville fractional integral using the trapezoidal method and Simpson's rule. In terms of examples, we provide various examples of fractional integrals with different functions and values of α and compare the errors and their error ratios. In comparison to the trapezoidal method, Simpson's method is better at reaching the provided fractional integral value, according to the comparison results. Furthermore, for $\alpha = 1$, the fractional integral of the periodic function converges quickly, but this does not necessarily apply to other alpha values.

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